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Exact Probability Distribution for Soluble Model with Quadratic Noise

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A one-dimensional evolution equation transformable into a linear one coupled to a quadratic Smoluchowski (an Ornstein–Uhlenbeck) noise is considered. A one-dimensional probability distribution is obtained by way of a characteristic function which is expressed by functionals of the Smoluchowski process. It is shown that in the frame of the presented approach the probability density can be found only for a particular value of the damping constant in the linear-type relaxation equation. It is also shown that in a special case the white noise limit may be performed.

KEY WORDS: Langevin equation; quadratic noise; functionals of stochastic process; exactly solved models.

1. INTRODUCTION

In recent years we have seen an increasing interest in evolution equations with fluctuating parameters. The main reason is that equations of this type become more and more important for different applications.

The one-dimensional differential equation with one linear random parameter is the simplest case. This type of problem has been investigated in connection with instabilities of the system induced by noises.[1-3] A more difficult problem is when an evolution equation is a nonlinear function of the fluctuating parameter. This paper is concerned with a system described by a nonlinear ordinary differential equation with a quadratic noise assuming that this equation is transformable into a linear equation. In the considered case, the starting nonlinear equation is coupled to the multiplicative noise and the transformation leads to a linear one with the additive noise. Our primary purpose is to obtain the exact one-dimensional

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probability distribution for the process valid for all time $t > 0$. It is assumed that the initial probability density is the δ distribution and that the noise is a stationary Smoluchowski process. We are able to solve the problem only for particular values of parameters that occur in the evolution equation. And only in this case we can obtain a compact formula for the probability distribution.

There are several works on evolution equations of the Langevin type with a quadratic noise.[4–6] San Miguel and Sancho[4] derived an approximate form of a Fokker–Planck equation associated with the Langevin equation. Wódkiewicz[5] obtained formally the exact equation for the probability density. His equation is useful only for approximate calculations for two reasons. Firstly, it is an integro-differential equation, and secondly, the kernel of this equation is a differential operator with an infinite number of derivatives.

In the remainder of this paper, we proceed as follows. In Section 2 we present the model. In Section 3, the probability distribution related to the stochastic process is investigated by way of its characteristic function. It can be expressed by a functional of the Smoluchowski noise. Following Van Kampen[7] we define the "curtailed" functional which obeys a definite equation. Assuming that the solution of this equation may be represented by a Gaussian-type function, we obtain a set of three ordinary differential equations of the first order. One of these equations is the Riccati one. It turns out that explicit formulas for the general solution of the Riccati equation can be given only for particular values of parameters. It is shown in Section 4. In Section 5 we present the main result of the paper, namely, the probability density which is expressed by the Riemann integral of the real-valued function over a real positive half-axis. In Section 6 we present simple characteristics of the linear process with the quadratic noise and compare them with characteristics of the linear process coupled to the linear noise. The problem of the white noise limit is also considered. Section 7 contains final remarks.

2. PROBLEM

This paper deals with a one-dimensional ordinary differential (an evolution or a kinetic) equation coupled to a quadratic noise. An example of such a type of equation is given by

$$\dot{x}_t = a(x_t) + L^2 b(x_t) \tag{2.1}$$

where

$$x \in U \subset \mathbb{R}, \qquad x_{t_0} = x_0 \tag{2.2}$$

and L is a parameter which is the random variable

$$L = \lambda + y_t \tag{2.3}$$

Here, λ is the expectation value of L and y_t is a noise with the mean value

$$\langle y_t \rangle = 0 \tag{2.4}$$

Because y_t may not be a white noise, we take the simplest model of a non-white noise, namely, y_t is assumed to be a stationary Smoluchowski process

$$dy_t = -\alpha y_t\, dt + (2\gamma)^{1/2}\, dW_t \tag{2.5}$$

where

$$\alpha, \gamma \in \mathbb{R}^+, \qquad y \in \mathbb{R} \tag{2.6}$$

and subjects to given initial conditions

$$\langle y_0 \rangle = 0, \qquad \langle y_0^2 \rangle = \gamma/\alpha \tag{2.7}$$

Hence, y_t is a colored noise with (2.4) and

$$\langle y_t y_s \rangle = (\gamma/\alpha) \exp(-\alpha\, |t-s|) \tag{2.8}$$

On the whole, the problem set in this form is unsoluble, although the Kolmogorov–Fokker–Planck equation for the two-dimensional diffusion process (x_t, y_t) is known. The problem considerably simplifies when Eq. (2.1) can be transformed into a linear one, scil. if

$$\frac{d}{dx}\left[\frac{a(x)}{b(x)}\right] = \frac{c}{b(x)} > 0 \tag{2.9}$$

where c is a constant parameter. Then for the variable

$$z = \frac{a(x)}{b(x)} \tag{2.10}$$

we have

$$\dot{z}_t = cz_t + c(\lambda + y_t)^2 \tag{2.11}$$

Let us consider a slightly different equation

$$\dot{z}_t = cz_t + v(\lambda + y_t)^2 \tag{2.12}$$

with the new constant v.

Our aim is to determine the probability distribution $P(z, t)$ of the process z_t (2.12) with the following initial conditions:

$$z_{t_0} = z_0 \qquad \text{for} \quad t_0 = 0 \tag{2.13}$$

$$P_1(y, 0) = (\alpha/2\pi\gamma)^{1/2} \exp(-\alpha y^2/2\gamma) \tag{2.14}$$

where $P_1(y, 0)$ is the initial probability distribution for y_t (2.5). The method to be used is based on the characteristic function of z_t and certain functionals of the Smoluchowski process.

3. CHARACTERISTIC FUNCTION

Simple properties of the process z_t (2.12) can be investigated and obtained from its characteristic function

$$C(\omega, t) = \int_{-\infty}^{+\infty} dz\, e^{i\omega z} P(z, t) = \langle e^{i\omega z_t} \rangle \tag{3.1}$$

From Eq. (2.12) it follows that

$$C(\omega, t) = F[y_t \,|\, \omega, t] \exp[i\omega z_0 e^{ct} + i\omega\lambda^2 v(e^{ct} - 1)/c] \tag{3.2}$$

where the functional

$$F[y_t \,|\, \omega, t] = \left\langle \exp\left[i\omega v e^{ct} \int_0^t d\tau\, e^{-c\tau}(y_\tau^2 + 2\lambda y_\tau) \right] \right\rangle_1 \tag{3.3}$$

and the subscript 1 indicates that the mean value is taken over all realization of the Smoluchowski process y_t (2.5) with the initial value (2.14).

For fixed $t = T$ we define[8]

$$\Omega = \omega v e^{cT} \tag{3.4}$$

and

$$F_1[y_t \,|\, \Omega, T] = \left\langle \exp\left[i\Omega \int_0^T d\tau\, e^{-c\tau}(y_\tau^2 + 2\lambda y_\tau) \right] \right\rangle_1 \tag{3.5}$$

Then, of course,

$$F[y_t \,|\, \omega, t] = F_1[y_t \,|\, \Omega = \omega v e^{ct}, t] \tag{3.6}$$

Let us define the "curtailed" functional[7]

$$\Gamma(y, t) = \left\langle \delta(y_t - y) \exp\left[i\Omega \int_0^t d\tau\, e^{-c\tau}(y_\tau^2 + 2\lambda y_\tau) \right] \right\rangle_1 \tag{3.7}$$

where for notatinal abbreviation we have dropped the dependence upon y_t and Ω.

It can be shown[7] that

$$F_1[y_t | \Omega, t] = \int_{-\infty}^{+\infty} dy\, \Gamma(y, t) \tag{3.8}$$

and $\Gamma(y, t)$ fulfills the following equation:

$$\frac{\partial \Gamma(y, t)}{\partial t} = \left[\alpha \frac{\partial}{\partial y} y + \gamma \frac{\partial^2}{\partial y^2} + i\Omega e^{-ct}(y^2 + 2\lambda y) \right] \Gamma(y, t) \tag{3.9}$$

with the Cauchy boundary condition

$$\Gamma(y, 0) = P_1(y, 0) \tag{3.10}$$

Let us look for a solution of Eq. (3.9) in the form of the gaussian-type function

$$\Gamma(y, t) = \exp[A(t)\, y^2 + B(t)\, y + C(t)] \tag{3.11}$$

where $A(t)$, $B(t)$, and $C(t)$ are to be found. They obey the following differential equations:

$$\dot{A} = 4\gamma A^2 + 2\alpha A + i\Omega e^{-ct}, \qquad A(0) = -\alpha/2\gamma \tag{3.12}$$

$$\dot{B} = (\alpha + 4\gamma A)B + 2i\lambda\Omega e^{-ct}, \qquad B(0) = 0 \tag{3.13}$$

$$\dot{C} = \alpha + 2\gamma A + \gamma B^2, \qquad C(0) = (1/2)\ln(\alpha/2\pi\gamma) \tag{3.14}$$

It is seen that if the solution $A(t)$ of Eq. (3.12) is known, then Eqs. (3.13) and (3.14) can formally be integrated. However, Eq. (3.12) is a Riccati one and it is rather difficult to solve it.

4. SOLUTION OF RICCATI EQUATION

It is well known that any Riccati equation may be transformed into a homogeneous second-order linear differential equation. In the considered case, by the substitution

$$A = -\dot{X}/4\gamma X \tag{4.1}$$

Eq. (3.12) leads to

$$\ddot{X} - 2\alpha\dot{X} + 4i\gamma\Omega e^{-ct}X = 0 \tag{4.2}$$

The change of the independent variable

$$s = 4i\gamma\Omega e^{-ct} \tag{4.3}$$

transforms Eq. (4.2) into

$$s\ddot{X}(s) + \left(1 + \frac{2\alpha}{c}\right)\dot{X}(s) + \frac{1}{c^2}X(s) = 0 \tag{4.4}$$

An explicit solution of this equation is known for two cases.[9] If

$$c^2 = -1, \qquad 1 + \frac{2\alpha}{c} = -2n, \qquad n \in N \tag{4.5}$$

But then $\alpha = \pm i(2n+1)/2$ and this is in contradiction to the assumption (2.6). This case must be cast away. If

$$c = 4\alpha/(2n-1), \qquad n \in N \cup \{0\} \tag{4.6}$$

then two particular solutions of (4.4) read[9]

$$X_1(s) = \frac{d^n}{ds^n}\cos\left[\frac{(n-\frac{1}{2})\sqrt{s}}{\alpha}\right] \tag{4.7}$$

$$X_2(s) = \frac{d^n}{ds^n}\sin\left[\frac{(n-\frac{1}{2})\sqrt{s}}{\alpha}\right] \tag{4.8}$$

Hence, with the help of (4.3) and (4.1), two particular solutions of the Riccati equation (3.12) are known. The general solution can be found explicitly.

From Eq. (4.6) it follows that

$$c = -4\alpha < 0 \qquad \text{for} \quad n = 0 \tag{4.9}$$

and

$$c > 0 \qquad \text{for} \quad n \in N \tag{4.10}$$

In the deterministic case corresponding to (2.12) the solutions are unstable for (4.10) and stable for (4.9). Therefore, we will consider only relaxation problem, $c<0$ (4.9). In this case, the solution of Eq. (3.12) has the following form:

$$A(t)=\frac{\alpha}{2\gamma}r(t)\frac{M\sin r(t)+\cos r(t)}{M\cos r(t)-\sin r(t)} \tag{4.11}$$

where the notation

$$r(t)=r\exp(2\alpha t) \tag{4.12}$$

$$r^2=i\gamma\Omega/\alpha^2 \tag{4.13}$$

has been introduced and the constant parameter M is given by

$$M=\frac{\sin r-r\cos r}{\cos r+r\sin r} \tag{4.14}$$

Equation (3.13) can be solved, but the solution is expressed by the Fresnel integrals. If we simplify the model (2.12) setting

$$\lambda=0 \tag{4.15}$$

then

$$B(t)=0 \tag{4.16}$$

and

$$C(t)=C(0)+\alpha t+\frac{1}{2}\ln\frac{M\cos r-\sin r}{M\cos r(t)-\sin r(t)} \tag{4.17}$$

After the simplifications (4.9) and (4.15) our starting model reduces to the form

$$\dot{z}_t=-4\alpha z_t+\nu y_t^2 \tag{4.18}$$

Now, the explicit formula for the functional (3.5) can be calculated and reads

$$F_1[y_t\,|\,\Omega,t]=[\cos r(e^{2\alpha t}-1)-r\sin r(e^{2\alpha t}-1)]^{-1/2} \tag{4.19}$$

with r defined by Eq. (4.13).

5. PROBABILITY DISTRIBUTION

Using formula (4.19) and Eq. (3.6), from Eq. (3.2) for the case $\lambda=0$ one can obtain the characteristic function in the following form

$$C(\omega,t)=\exp(i\omega z_0 e^{-4\alpha t})[\cos(r_0\sqrt{\omega}\,(1-e^{-2\alpha t})) - r_0\sqrt{\omega}\,e^{-2\alpha t}\sin(r_0\sqrt{\omega}\,(1-e^{-2\alpha t}))]^{-1/2} \tag{5.1}$$

where

$$r_0^2 = i\gamma\nu/\alpha^2 \tag{5.2}$$

The final result for the one-dimensional probability distribution of the process (4.18) with initial condition (2.13) is

$$P(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \times \frac{\exp[-i\omega(z - z_0 e^{-4\alpha t})]}{[\cos(r_0\sqrt{\omega}(1 - e^{-2\alpha t})) - r_0\sqrt{\omega}\, e^{-2\alpha t} \sin(r_0\sqrt{\omega}(1 - e^{-2\alpha t}))]^{1/2}} \tag{5.3}$$

For $t = 0$, obviously $P(z, 0) = \delta(z - z_0)$. The stationary state of the process (4.18) exists for all assumed values of parameters α, γ (2.6) and $\nu \in \mathbb{R}$. The stationary probability density

$$P_{\mathrm{st}}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega z}}{[\cos(r_0\sqrt{\omega})]^{1/2}} \tag{5.4}$$

does not depend upon the initial value of the considered process. In such a case we say that the process is ergodic (see Section 10 in Ref. 10).

The knowledge of the characteristic function $C(\omega, t)$ is sufficient to calculate the main characteristics of the linear process (as the mean value, fluctuations, and so on). For nonlinear model (2.1) with (2.9), (4.9), and (4.15) we need

$$P(x, t) = \frac{c}{b(x)} P\left(z = \frac{a(x)}{b(x)}, t\right) \tag{5.9}$$

and with the help of (5.3), $P(x, t)$ is known.

6. SIMPLE CHARACTERISTICS OF THE LINEAR PROCESS

The moments $\langle z_t^n \rangle$, $n \in N$, of the linear process (2.11) can be calculated directly from Eq. (2.11). For the model (4.18) we can utilize the characteristic function (5.1). The average value and the fluctuations of the process (4.18) are given by

$$\langle z_t \rangle = z_0 e^{-4\alpha t} + \frac{\gamma\nu}{4\alpha^2}(1 - e^{-4\alpha t}) \tag{6.1}$$

$$\langle z_t^2 \rangle - \langle z_t \rangle^2 = \frac{\gamma^2\nu^2}{12\alpha^4}(1 - 4e^{-6\alpha t} + 3e^{-8\alpha t}) \tag{6.2}$$

For the corresponding model coupled to the linear noise

$$\dot{z}_t = -4\alpha z_t + v y_t \tag{6.3}$$

one gets

$$\langle z_t \rangle = z_0 e^{-4\alpha t}$$

$$\langle z_t^2 \rangle - \langle z_t \rangle^2 = \frac{\gamma v^2}{60\alpha^3} (3 - 8e^{-5\alpha t} + 5e^{-8\alpha t}) \tag{6.5}$$

In general, the white noise limit may not be carried out for models with a nonlinear noise. Because of some peculiarity of the model (4.18), let us consider the problem of the white noise limit. By the substitution

$$\gamma = \alpha^2 \sigma^2 / 2 \tag{6.6}$$

the limit $\alpha \to \infty$ corresponds to the case of the white noise in Eq. (2.5), $y_t\, dt = \sigma\, dW_t$. It can be seen from Eqs. (6.1) and (6.2) that if (i) $v = c = -4\alpha$ as in Eq. (2.11), then the white noise limit may not be performed (all moments diverge), and (ii) if v is independent of α then the limit $\alpha \to \infty$ is finite in all expressions and from Eqs. (6.1) and (6.2) one can obtain

$$\lim_{\alpha \to \infty} \langle z_t \rangle = v\sigma^2/8 \tag{6.7}$$

$$\lim_{\alpha \to \infty} (\langle z_t^2 \rangle - \langle z_t \rangle^2) = v^2 \sigma^4 / 48 \tag{6.8}$$

In this limit $\alpha \to \infty$, the characteristic function (5.1) becomes

$$C(\omega, t) = C(\omega) = [\cos(iv\sigma^2\omega/2)^{1/2}]^{-1/2} \tag{6.9}$$

and the probability distribution $P(z, t) = P(z)$ (5.3) is well defined.

7. FINAL REMARKS

We have described the method for solving the simple model with a quadratic noise. We have started from a general linear model (2.12) which contains three free parameters, c, λ, and v. Our approach gives the exact probability density of the process only when $c = -4\alpha$, $\lambda = 0$ and for arbitrary v. The model of interest has a particular form since the damping parameter in the relaxation problem depends linearly on the inversion of the correlation time of the noise.

In the case of the linear model (4.18) the probability density is known; however, it is useless for calculation of the moments. On the contrary for nonlinear models transformable into linear ones the form of the probability distribution must be known.

In a previous paper[3] the example of a nonlinear model transformable into a linear one coupled to the linear noise was presented. This model, which is stable in the deterministic case, exhibits instabilities induced by the noise, even if $c = -4\alpha$, as in the present paper. It would be interesting to compare this model with the analogical one coupled to the quadratic noise. To do this the determination of the most probable values of the process, which correspond to the extrema of $P(x, t)$ (5.9), should be carried out. Such a numerical analysis is being performed.

It is remarkable that taking of the white noise limit for the model with the quadratic noise is possible. It seems that in this limit the process z_t can be treated as some kind of transformation of the white noise. Up to now our efforts to determine the form of this transformation have turned out fruitless.

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